Axial Rod Stretching

Setup and assumptions

A straight rod of circular cross-section (diameter d) and length L is fixed at $x = 0$ and pulled by an axial force F at $x = L$. The material is linear elastic with Young's modulus E; strains are small. Let $u(x)$ be the axial displacement of the cross-section at x.

Equilibrium

Cut the rod at any x: axial equilibrium of the free piece requires the internal normal force to equal the applied load, $N(x) = F$, at every section. Assuming the stress is uniform over the cross-section,

$$\sigma \;=\; \frac{N}{A} \;=\; \frac{F}{A}, \qquad A \;=\; \frac{\pi d^2}{4}. \tag{1}$$

Kinematics and constitutive law

The axial strain is the displacement gradient, and Hooke's law relates it to stress:

$$\varepsilon \;=\; \frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \sigma \;=\; E\,\varepsilon. \tag{2}$$

Governing equation

Combining the three ingredients, $N = EA\,u'$, and equilibrium with no distributed axial load gives $\mathrm{d}N/\mathrm{d}x = 0$, i.e.

$$EA\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \;=\; 0, \qquad u(0) = 0, \qquad EA\,u'(L) = F. \tag{3}$$

This is the axial analogue of the fourth-order bending equation — here the problem is second order. Integrating twice: $u = C_1 x + C_2$; the boundary conditions give $C_2 = 0$ and $C_1 = F/EA$.

Solution

$$\boxed{u(x) \;=\; \frac{Fx}{EA}} \qquad \Rightarrow \qquad \varepsilon \;=\; \frac{F}{EA} \;=\; \frac{\sigma}{E}, \qquad \delta \;=\; u(L) \;=\; \frac{FL}{EA}. \tag{4}$$

Strain and stress are uniform along the rod; the elongation is linear in load and length, and inversely proportional to stiffness and area.

Worked example

$E = 10$ MPa (a soft elastomer), $d = 10$ mm, $L = 500$ mm, $F = 50$ N:

$$A = \frac{\pi \times 10^2}{4} = 78.5\ \mathrm{mm}^2, \qquad \sigma = \frac{50}{78.5 \times 10^{-6}} = 0.637\ \mathrm{MPa}, \qquad \varepsilon = 6.37\%, \qquad \delta = 31.8\ \mathrm{mm}. \tag{5}$$

Set the simulator sliders to these values and compare. Then switch E to steel (200 GPa): the same load produces $\delta = 1.6\ \mu$m — the drawing does not visibly move, and that is the point. Beyond roughly 10% strain the small-strain, linear-elastic assumptions degrade.