Euler Buckling

Setup and assumptions

A straight prismatic member of length L and rectangular cross-section $w \times h$ carries an axial compressive force P; the material is linear elastic with Young's modulus E. We address the question of "at what load does the straight configuration stop being the only equilibrium?". This is an *eigenvalue* problem.

Weak axis

The rectangle has two centroidal second moments, $I_{xx} = wh^3/12$ and $I_{yy} = hw^3/12$, for bending in the h- and w-directions respectively. A compressed member bows whichever way is easiest — about the weak axis, $I_{\min} = \min(I_{xx}, I_{yy})$: a ruler squashed end-to-end always bows in its thin direction.

Governing equation (pinned–pinned)

Suppose the member has deflected laterally by a small $u(x)$. Moment equilibrium of the segment beyond a cut at x, taken about the cut, shows the axial force now produces an internal bending moment $M(x) = -P\,u(x)$. With moment–curvature $M = EI\,u''$,

$$EI\,u'' + P\,u \;=\; 0 \qquad \Longrightarrow \qquad u'' + k^2 u = 0, \quad k^2 = \frac{P}{EI}, \tag{1}$$

with $u(0) = u(L) = 0$ for pinned ends. The general solution is $u = A\sin kx + B\cos kx$; the condition $u(0) = 0$ gives $B = 0$, and $u(L) = 0$ then requires $A\sin kL = 0$. Either $A = 0$ (the straight state — always an equilibrium) or $\sin kL = 0$, i.e. $kL = n\pi$. A bent equilibrium first becomes possible at $n = 1$:

$$\boxed{P_{\text{cr}} \;=\; \frac{\pi^2 EI_{\min}}{L^2}} \qquad \text{with mode shape} \qquad u(x) = A\sin\frac{\pi x}{L}. \tag{2}$$

The amplitude A is *indeterminate*: linearised theory predicts the critical load, not the actual displacement.

Other end conditions

With general supports the governing form is fourth order, $EI\,u'''' + P\,u'' = 0$, with solution $u = A\sin kx + B\cos kx + Cx + D$ fitted to the four end conditions. Each case reduces to the pinned–pinned result with an *effective length* KL — the distance between inflection points of its mode: $P_{\text{cr}} = \pi^2 EI_{\min}/(KL)^2$.

End conditions	Characteristic equation	K
pinned–pinned	$\sin kL = 0$	1
fixed–free	$\cos kL = 0$	2
fixed–pinned	$\tan kL = kL \quad (kL = 4.493)$	$\pi/4.493 = 0.699$
fixed–fixed	$\cos kL = 1$	1/2

Clamping the ends quadruples the capacity; a flagpole geometry carries only a quarter.

Worked example (simulator defaults)

$E = 3$ GPa (acrylic), $w = 8$ mm, $h = 25$ mm, $L = 500$ mm, pinned–pinned. The weak axis is y–y: $I_{yy} = 25 \times 8^3/12 = 1067\ \text{mm}^4$, well below $I_{xx} = 10\,417\ \text{mm}^4$. Then

$$P_{\text{cr}} = \frac{\pi^2 \times 3 \times 10^9 \times 1.067 \times 10^{-9}}{0.5^2} \approx 126\ \text{N}, \tag{3}$$

about 13 kg — try it with an acrylic strip. Fixed–free drops it to ≈ 32 N, fixed–fixed raises it to ≈ 505 N. The stress at buckling (≈ 0.63 MPa) is far below yield, so the elastic analysis governs; stocky columns yield first, and post-buckling behavior is beyond this linearised model.