Cantilever Beam Bending

Setup and assumptions

A prismatic cantilever of length L and rectangular cross-section $w \times h$ (width w, height h) is clamped at $x = 0$ and carries a transverse point load F at the free end $x = L$. The material is linear elastic with Young's modulus E. We measure the deflection $u(x)$ positive in the direction of the load and adopt the Euler–Bernoulli assumptions: cross-sections remain plane and perpendicular to the beam axis (shear deformation neglected), and deflections are small ($\delta \lesssim 0.1\,L$).

Section property

The second moment of area of the rectangle about its horizontal centroidal axis is

$$I \;=\; \int_A y^2\,\mathrm{d}A \;=\; w\int_{-h/2}^{h/2} y^2\,\mathrm{d}y \;=\; \frac{wh^3}{12}. \tag{1}$$

Governing equation

Moment–curvature (from plane sections and Hooke's law) gives $M = EI\,u''$. Equilibrium of a beam slice gives $\mathrm{d}M/\mathrm{d}x = V$ and $\mathrm{d}V/\mathrm{d}x = -q$. Between loads $q = 0$, so

$$EI\,\frac{\mathrm{d}^4u}{\mathrm{d}x^4} \;=\; 0. \tag{2}$$

Integrating four times introduces four constants:

$$u''' = C_1, \qquad u'' = C_1x + C_2, \qquad u' = \tfrac{1}{2}C_1x^2 + C_2x + C_3, \qquad u = \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2 + C_3x + C_4. \tag{3}$$

Boundary conditions

The clamped end can neither deflect nor rotate; the free end carries no bending moment, and its shear force equals the applied load:

$$u(0) = 0, \qquad u'(0) = 0, \qquad EI\,u''(L) = 0, \qquad EI\,u'''(L) = -F. \tag{4}$$

The first two give $C_4 = C_3 = 0$; the last two give $C_1 = -F/EI$ and $C_2 = FL/EI$.

Solution

$$\boxed{u(x) \;=\; \frac{Fx^2\,(3L - x)}{6\,EI}} \qquad \Rightarrow \qquad \delta \;=\; u(L) \;=\; \frac{FL^3}{3\,EI}, \qquad \theta_L \;=\; u'(L) \;=\; \frac{FL^2}{2\,EI}. \tag{5}$$

The bending moment is largest at the wall, $M_{\max} = FL$, so the peak bending stress (at the outer fibres, $y = \pm h/2$) is

$$\sigma_{\max} \;=\; \frac{M_{\max}\,(h/2)}{I} \;=\; \frac{6\,FL}{wh^2}. \tag{6}$$

Deflection grows with L^3 and shrinks with E and h^3: doubling the height is eight times stiffer, doubling the width only twice.

Worked example (simulator defaults)

$E = 2$ GPa, $L = 500$ mm, $w = 25$ mm, $h = 10$ mm, $F = 4$ N:

$$I = \frac{25 \times 10^3}{12} = 2083\ \mathrm{mm}^4, \qquad \delta = \frac{4 \times 0.5^3}{3 \times (2 \times 10^9) \times (2.083 \times 10^{-9})} \approx 40\ \mathrm{mm}, \tag{7}$$

with $\sigma_{\max} = 4.8$ MPa. Set the simulator sliders to these values and compare.