SOLID MECHANICS

# Cylinder buckling: derivation

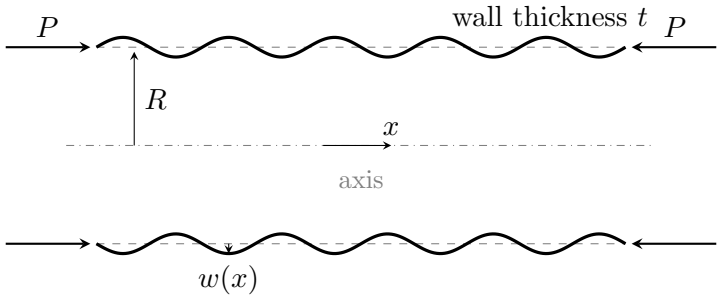
Axisymmetric buckling of a thin cylinder under axial compression

## 1 The problem

A long, thin circular cylinder of radius $R$ (measured to the mid-surface) and wall thickness $t$ is compressed along its axis. A total force $P$ acts on each end, spread uniformly around the circumference. The compressive stress in the wall before buckling is

$$\sigma = \frac{P}{2\pi R\, t} \qquad \text{(compression positive)}, \tag{1}$$

and the axial force per unit length of circumference is $N_x = -\sigma t$.




Section through the axis. Dashed lines: the wall before buckling. The buckle is the same at every angle around the circumference.

The coordinates are $x$ along the axis, $\theta$ around the circumference, and $z$ through the wall, measured from the mid-surface. The deformation is axisymmetric: the axial displacement $u(x)$ and the radial displacement $w(x)$, positive outward, depend only on $x$, and there is no circumferential displacement.

**Assumptions.**

- The wall is thin ($t$ much smaller than $R$), and normals to the mid-surface remain straight and normal to it (Kirchhoff–Love kinematics).
- The material is linear elastic and isotropic, with Young's modulus $E$ and Poisson's ratio $\nu$.
- The buckle is short compared with the radius (shallow-shell, or Donnell, approximation).
- Rotations are moderate: only the slope $\mathrm{d}w/\mathrm{d}x$ enters the nonlinear terms.
- The cylinder is long compared with the buckle, so the end conditions do not affect the critical load (Section 7 treats a finite length).

The derivation has three parts, as for the column and the plate: kinematics, compatibility and equilibrium. The material law then combines them.

## 2 Kinematics

**Axial strain.** A line element along the axis stretches, and its slope lengthens it further:

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2. \tag{2}$$

**Hoop strain.** An outward displacement $w$ changes the circumference from $2\pi R$ to $2\pi(R+w)$:

$$\varepsilon_\theta = \frac{w}{R}. \tag{3}$$

**Change of curvature.** Along the axis, $\kappa_x = -\mathrm{d}^2w/\mathrm{d}x^2$. The hoop curvature changes from $1/R$ to $1/(R+w)$, a change of about $-w/R^2$. If the buckle has a length scale $\ell$, this is smaller than $\kappa_x$ by a factor of order $\ell^2/R^2$, and it is neglected (shallow-shell approximation).

The strain at a distance $z$ from the mid-surface is $\varepsilon_x + z\,\kappa_x$ along the axis.

For a column, the same steps give $\varepsilon = \mathrm{d}u/\mathrm{d}x + \frac{1}{2}(\mathrm{d}w/\mathrm{d}x)^2$ and $\kappa = -\mathrm{d}^2w/\mathrm{d}x^2$, with no hoop strain. For a flat plate, the term $w/R$ is absent.

## 3 Compatibility

For a general deformation of the cylinder wall, with $y = R\theta$ the arc length around the circumference, the three membrane strains must satisfy Donnell's compatibility condition

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\,\frac{\partial^2 w}{\partial x^2}. \tag{4}$$

For an axisymmetric deformation, all derivatives with respect to $y$ vanish and $\varepsilon_y = \varepsilon_\theta$, so the condition reduces to

$$\frac{\mathrm{d}^2\varepsilon_\theta}{\mathrm{d}x^2} = \frac{1}{R}\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}. \tag{5}$$

Integrating twice, with no circumferential displacement, gives $\varepsilon_\theta = w/R$, which is (3). The hoop strain is therefore not independent: it is set by the radial displacement. This is how the curvature of the wall couples bending to stretching.

For a column, there is no compatibility condition. For a flat plate, the term with $1/R$ is absent, and the in-plane strains are not tied to $w$ at linear order.

## 4 Equilibrium

On an element of length $\mathrm{d}x$ along the axis and $R\,\mathrm{d}\theta$ around the circumference, the forces and moment per unit length are the axial force $N_x$, the hoop force $N_\theta$, the transverse shear force $Q_x$ and the bending moment $M_x$.

**Axial force balance.**

$$\frac{\mathrm{d}N_x}{\mathrm{d}x} = 0 \quad\Longrightarrow\quad N_x = -\sigma t \ \text{ everywhere.} \tag{6}$$

**Moment balance.**

$$Q_x = \frac{\mathrm{d}M_x}{\mathrm{d}x}. \tag{7}$$

**Radial force balance** (outward positive). The shear force, the axial force acting on the sloped wall and the hoop force acting on the curved wall must balance:

$$\frac{\mathrm{d}Q_x}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(N_x \frac{\mathrm{d}w}{\mathrm{d}x}\right) - \frac{N_\theta}{R} = 0. \tag{8}$$

The hoop force $N_\theta$ acts tangentially at both sides of the element; because the wall is curved, the two forces have a resultant $N_\theta/R$ per unit area, directed toward the axis when $N_\theta$ is tensile. Using $\mathrm{d}N_x/\mathrm{d}x = 0$,

$$\frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} + N_x \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{N_\theta}{R} = 0. \tag{9}$$

For a column, the same balance gives $\mathrm{d}N/\mathrm{d}x = 0$ and $\mathrm{d}^2M/\mathrm{d}x^2 + N\,\mathrm{d}^2w/\mathrm{d}x^2 + q = 0$. For a flat plate, the term $N_\theta/R$ is absent.

## 5 Material law and governing equation

For a thin wall in plane stress,

$$N_x = \frac{Et}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\theta), \qquad N_\theta = \frac{Et}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_x), \qquad M_x = D\,\kappa_x = -D\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{10}$$

with the bending stiffness $D = Et^3/[12(1-\nu^2)]$.

**Hoop force.** The axial force is fixed, $N_x = -\sigma t$. Eliminating $\varepsilon_x$ between the two membrane relations and using (3),

$$N_\theta = Et\,\varepsilon_\theta + \nu N_x = \frac{Et}{R}\,w - \nu\sigma t. \tag{11}$$

**Governing equation.** Substituting $M_x$, $N_x$ and $N_\theta$ into (9) and changing the sign,

$$D\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \sigma t\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{Et}{R^2}\,w = \frac{\nu\sigma t}{R}. \tag{12}$$

**State before buckling.** A uniform displacement $w_0 = \nu\sigma R/E$ satisfies (12). It is the outward expansion caused by Poisson's ratio, and it gives $N_\theta = 0$, as expected for a cylinder loaded only along its axis.

## 6 Linearisation for buckling

Write $w = w_0 + w_1$, where $w_1$ is the buckling displacement. Subtracting the equation for $w_0$ from (12),

$$D\,\frac{\mathrm{d}^4 w_1}{\mathrm{d}x^4} + \sigma t\,\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} + \frac{Et}{R^2}\,w_1 = 0. \tag{13}$$

This equation is linear in $w_1$ without further approximation, because $N_x$ is fixed by axial equilibrium; the quadratic term in $\varepsilon_x$ changes only the axial displacement $u$. Equation (13) is the equation of a beam of unit width on an elastic foundation of stiffness $Et/R^2$, compressed by the force $\sigma t$. It is the equation used in the lecture.

## 7 Boundary conditions

**Long cylinder.** The ends are far from most of the cylinder, so we look for solutions that are periodic along $x$.

**Finite length $L$ with simply supported ends.** The radial displacement and the bending moment vanish at the ends, while the ends remain free to move along the axis so that they can carry the load:

$$w_1 = \frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} = 0 \quad \text{at } x = 0 \text{ and } x = L. \tag{14}$$

## 8 Solution

**Characteristic equation.** Substituting $w_1 = A\, e^{ikx}$ into (13) gives

$$Dk^4 - \sigma t\, k^2 + \frac{Et}{R^2} = 0 \qquad \Longrightarrow \qquad k^2 = \frac{\sigma t \pm \sqrt{\sigma^2 t^2 - 4DEt/R^2}}{2D}. \tag{15}$$

- If $\sigma t < 2\sqrt{DEt}/R$, the roots $k$ are complex. The solutions are oscillations that grow or decay along the axis over a distance of order $\sqrt{Rt}$. They describe disturbances near the ends, not a buckle along the cylinder: no periodic solution exists, and the cylinder stays straight.
- At $\sigma t = 2\sqrt{DEt}/R$, the two roots coincide at a real value $k_c$, and a periodic solution $w_1 = A \sin k_c x$ becomes possible. This is the critical stress.

Equivalently, a periodic buckle of wave number $k$ is in equilibrium at the stress

$$\sigma t = Dk^2 + \frac{Et}{R^2 k^2}. \tag{16}$$

The first term is the bending resistance, which is large for short buckles; the second is the resistance from hoop stretching, which is large for long buckles.

**Finite length.** The functions $w_1 = A \sin(m\pi x/L)$, $m = 1, 2, 3, \ldots$, satisfy the end conditions. Each gives an eigenvalue from (16) with $k = m\pi/L$. When $L$ is much larger than $\sqrt{Rt}$, some integer $m$ gives $m\pi/L$ very close to $k_c$, and the lowest eigenvalue is very close to the critical stress of the long cylinder.

## 9 Critical stress, buckle length and critical load

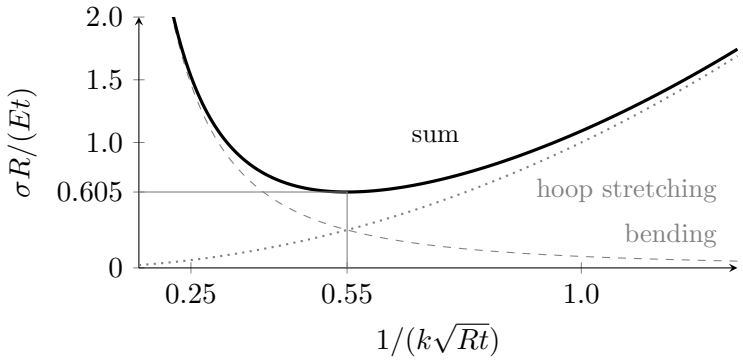
Minimising (16) over $k$ gives

$$k_c^4 = \frac{Et}{DR^2} = \frac{12(1-\nu^2)}{R^2 t^2}, \qquad \sigma_{cr} = \frac{2\sqrt{D\,Et}}{R\,t} = \frac{E\,t}{R\sqrt{3(1-\nu^2)}} \approx 0.605\,\frac{E\,t}{R} \quad (\nu = 0.3). \tag{17}$$

The half-wavelength of the critical buckle and the critical load are

$$\frac{\pi}{k_c} = \frac{\pi\sqrt{R\,t}}{[12(1-\nu^2)]^{1/4}} \approx 1.73\sqrt{R\,t} \quad (\nu = 0.3), \qquad P_{cr} = 2\pi R\,t\,\sigma_{cr} = \frac{2\pi E\,t^2}{\sqrt{3(1-\nu^2)}}. \tag{18}$$

The critical load does not depend on the radius. The cylinder stays straight for $P < P_{cr}$ and can take the buckled shape for $P \geq P_{cr}$.




Equation (16) for $\nu = 0.3$. The minimum is the critical stress.

## 10 Example

An aluminium cylinder with $E = 70$ GPa, $\nu = 0.3$, $R = 150$ mm, $t = 0.5$ mm and $L = 450$ mm:

- $R/t = 300$ and $\sqrt{3(1-\nu^2)} = 1.652$.
- $\sigma_{cr} = \dfrac{70 \times 10^9 \times (0.5/150)}{1.652} = 141$ MPa.
- $P_{cr} = 2\pi \times 0.15 \times 0.0005 \times 141 \times 10^6 = 66.5$ kN.
- Half-wavelength $= 1.73\sqrt{150 \times 0.5}$ mm $= 15.0$ mm, so the length holds 30 half-waves.
- 141 MPa is below the yield stress of common structural aluminium alloys such as 6061-T6, so the elastic analysis applies.

These are the default values of the cylinder buckling applet.

## 11 Remarks

1. **Other buckling shapes.** With the full Donnell equations, a buckle of the form $w = A\cos(kx)\cos(ly)$ has the same critical stress $\sigma_{cr}$ for every pair of wave numbers on the circle $k^2 + l^2 = k\,k_c$ (the Koiter circle). The axisymmetric mode is the point $l = 0$. Many buckling shapes therefore share the same critical load.

2. **Imperfection sensitivity.** The load falls steeply after buckling, so small departures from the perfect shape lower the buckling load substantially. Tests on axially compressed cylinders give loads well below $P_{cr}$, with wide scatter. Design practice multiplies the classical load by an empirical knockdown factor; NASA SP-8007 gives $\gamma = 1 - 0.901\,(1 - e^{-\phi})$ with $\phi = \frac{1}{16}\sqrt{R/t}$. For the example, $\phi = 1.08$ and $\gamma = 0.40$, so the design load is about 27 kN.

3. **Range of validity.** The shallow-shell approximation requires the buckle to be short compared with the radius: $1/k_c = 0.55\sqrt{Rt}$ for $\nu = 0.3$, which is $0.032\,R$ in the example. End conditions that restrain the Poisson expansion cause bending near the ends before buckling, which changes the classical load somewhat; this analysis ignores that effect. If $\sigma_{cr}$ exceeds the yield stress, the cylinder yields before it buckles elastically, and the result does not apply.