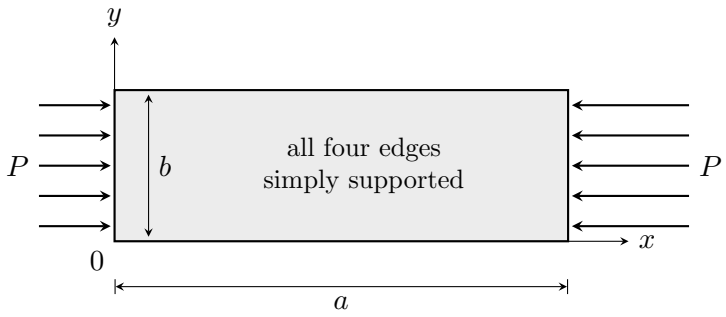
SOLID MECHANICS

Plate buckling: derivation

Buckling of a simply supported rectangular plate under uniaxial compression

1 The problem

A rectangular plate of length a (the loaded length), width b and thickness t is compressed along its length. A total force P acts on each of the two loaded edges $x = 0$ and $x = a$, spread uniformly over the edge. All four edges are simply supported: they cannot move out of the plane, but they are free to rotate.



The coordinates are x along the loaded length ($0 \le x \le a$), y across the width ($0 \le y \le b$), and z through the thickness, measured from the mid-surface. The out-of-plane deflection of the mid-surface is $w(x, y)$, positive in the $+z$ direction.

The compressive stress in the plate before buckling is

$$\sigma = \frac{P}{b\,t} \qquad \text{(compression positive)}, \tag{1}$$

and the corresponding in-plane force per unit width is $N_x = -\sigma t$.

Assumptions.

- The plate is thin (t much smaller than a and b), and normals to the mid-surface remain straight and normal to it (Kirchhoff plate theory).
- The material is linear elastic and isotropic, with Young's modulus E and Poisson's ratio ν.
- Rotations are moderate: only the slopes $\partial w/\partial x$ and $\partial w/\partial y$ enter the nonlinear terms (von Kármán plate theory).
- Before buckling, the plate is flat ($w = 0$) and carries the uniform force $N_x = -\sigma t$, with $N_y = N_{xy} = 0$.

The derivation has three parts, as for the column and the cylindrical shell: kinematics, compatibility and equilibrium. The material law then combines them.

2 Kinematics

Because normals remain straight and normal, the in-plane displacements through the thickness are

$$u(x,y,z) = u_0 - z\,\frac{\partial w}{\partial x}, \qquad v(x,y,z) = v_0 - z\,\frac{\partial w}{\partial y}, \tag{2}$$

where u_0 and v_0 are the displacements of the mid-surface. The strains are therefore a membrane part plus a part proportional to z:

$$\varepsilon_x = \varepsilon_x^0 + z\,\kappa_x, \qquad \varepsilon_y = \varepsilon_y^0 + z\,\kappa_y, \qquad \gamma_{xy} = \gamma_{xy}^0 + 2z\,\kappa_{xy}. \tag{3}$$

Membrane strains. The stretching of a line element includes the effect of its slope:

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_y^0 = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad \gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}. \tag{4}$$

Changes of curvature.

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -\frac{\partial^2 w}{\partial x\,\partial y}. \tag{5}$$

For a column, the same two steps give $\varepsilon^0 = \mathrm{d}u_0/\mathrm{d}x + \frac{1}{2}(\mathrm{d}w/\mathrm{d}x)^2$ and $\kappa = -\mathrm{d}^2w/\mathrm{d}x^2$. For a cylindrical shell of radius R, the hoop strain gains the term w/R.

3 Compatibility

The three membrane strains in (4) come from only two displacements, u_0 and v_0, so they cannot be chosen independently. Taking second derivatives and combining them so that u_0 and v_0 cancel,

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial u_0}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v_0}{\partial y}\right) - \frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) = 0, \tag{6}$$

leaves the compatibility condition

$$\frac{\partial^2 \varepsilon_x^0}{\partial y^2} + \frac{\partial^2 \varepsilon_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x\,\partial y} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}. \tag{7}$$

The right-hand side is quadratic in w. It matters after buckling, but it does not affect the critical load (Section 6). For a column there is no compatibility condition, because one strain and one curvature can always be integrated to give u_0 and w. For a cylindrical shell, the right-hand side gains the linear term $(1/R)\,\partial^2 w/\partial x^2$.

4 Equilibrium

The membrane forces, bending moments and transverse shear forces per unit length of edge are

$$(N_x, N_y, N_{xy}) = \int_{-t/2}^{t/2} (\sigma_x, \sigma_y, \tau_{xy})\,dz, \qquad (M_x, M_y, M_{xy}) = \int_{-t/2}^{t/2} (\sigma_x, \sigma_y, \tau_{xy})\,z\,dz,$$

$$(Q_x, Q_y) = \int_{-t/2}^{t/2} (\tau_{xz}, \tau_{yz})\,dz. \tag{8}$$

In-plane force balance.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \tag{9}$$

Moment balance about the y and x axes gives the transverse shear forces:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \qquad Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}. \tag{10}$$

Force balance normal to the plate. On the deflected element, the in-plane forces have components in the z direction, $N_x\,\partial w/\partial x + N_{xy}\,\partial w/\partial y$ on a face normal to x, and similarly on a face normal to y. With a pressure p in the $+z$ direction,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}\right) + p = 0. \tag{11}$$

Using (9) to remove the derivatives of the N terms, and substituting for Q_x and Q_y,

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x\,\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\,\frac{\partial^2 w}{\partial x\,\partial y} + N_y\,\frac{\partial^2 w}{\partial y^2} + p = 0. \tag{12}$$

For a column, the same balance gives $\mathrm{d}N/\mathrm{d}x = 0$ and $\mathrm{d}^2M/\mathrm{d}x^2 + N\,\mathrm{d}^2w/\mathrm{d}x^2 + q = 0$. For a cylindrical shell, (12) gains the term $-N_y/R$ from the hoop force acting on the curved wall.

5 Material law and governing equations

For a thin plate in plane stress, $\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$, $\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$ and $\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$. Integrating through the thickness,

$$N_x = \frac{Et}{1-\nu^2}\,(\varepsilon_x^0 + \nu\varepsilon_y^0), \quad N_y = \frac{Et}{1-\nu^2}\,(\varepsilon_y^0 + \nu\varepsilon_x^0), \quad N_{xy} = \frac{Et}{2(1+\nu)}\,\gamma_{xy}^0, \tag{13}$$

$$M_x = D\,(\kappa_x + \nu\kappa_y), \quad M_y = D\,(\kappa_y + \nu\kappa_x), \quad M_{xy} = D\,(1-\nu)\,\kappa_{xy}, \tag{14}$$

with the bending stiffness

$$D = \frac{E}{1-\nu^2}\int_{-t/2}^{t/2} z^2\,dz = \frac{E\,t^3}{12(1-\nu^2)}. \tag{15}$$

Moments. Substituting the curvatures, the moment terms in (12) become

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -D\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = -D\,\nabla^4 w. \tag{16}$$

Stress function. The in-plane equilibrium equations (9) are satisfied exactly by

$$N_x = \frac{\partial^2 F}{\partial y^2}, \qquad N_y = \frac{\partial^2 F}{\partial x^2}, \qquad N_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}. \tag{17}$$

Writing the membrane strains in terms of F with the material law, the left-hand side of the compatibility condition (7) equals $\nabla^4 F/(Et)$.

Governing equations (von Kármán plate equations).

$$D\,\nabla^4 w = \frac{\partial^2 F}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + p, \tag{18}$$

$$\frac{1}{Et}\,\nabla^4 F = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}. \tag{19}$$

6 Linearisation for buckling

Before buckling, $w_0 = 0$ and $F_0 = -\frac{1}{2}\,\sigma t\,y^2$, which gives $N_x = -\sigma t$ and $N_y = N_{xy} = 0$. Let w and $F = F_0 + f$ describe a small departure from this state, with $p = 0$, and keep only terms linear in w and f:

$$D\,\nabla^4 w + \sigma t\,\frac{\partial^2 w}{\partial x^2} = 0, \qquad \nabla^4 f = 0. \tag{20}$$

The two equations are uncoupled. The in-plane perturbation f does not affect buckling, and the critical load follows from the first equation alone. This is the plate equation used in the lecture. (For a cylindrical shell, the term w/R in the hoop strain couples w and f even at this linear order.)

7 Boundary conditions

On a simply supported edge, the deflection and the bending moment normal to the edge are zero. Along the edge $x = 0$, $w = 0$ for all y, so $\partial^2 w/\partial y^2 = 0$ there as well, and $M_x = -D(\partial^2 w/\partial x^2 + \nu\,\partial^2 w/\partial y^2) = 0$ reduces to $\partial^2 w/\partial x^2 = 0$. The four edges therefore give

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \;\text{ at } x = 0,\; a; \qquad w = \frac{\partial^2 w}{\partial y^2} = 0 \;\text{ at } y = 0,\; b. \tag{21}$$

8 Solution

Trial solution. Every term of the form

$$w = A\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}, \qquad m, n = 1, 2, 3, \ldots \tag{22}$$

satisfies all four boundary conditions. Here m is the number of half-waves along the loaded length and n the number across the width.

Substitution. With $\alpha = m\pi/a$ and $\beta = n\pi/b$, the derivatives are $\nabla^4 w = (\alpha^2 + \beta^2)^2\,w$ and $\partial^2 w/\partial x^2 = -\alpha^2 w$. Equation (20) becomes

$$\left[D\,(\alpha^2 + \beta^2)^2 - \sigma t\,\alpha^2\right] A = 0. \tag{23}$$

Eigenvalue condition. For most values of σ, the only solution is $A = 0$: the plate stays flat. A non-zero A is possible only when the bracket vanishes:

$$\sigma = \frac{D\,(\alpha^2 + \beta^2)^2}{t\,\alpha^2} = \frac{\pi^2 D}{b^2 t}\left(\frac{mb}{a} + \frac{n^2 a}{mb}\right)^2. \tag{24}$$

These stresses are the eigenvalues, and the corresponding shapes are the buckling modes. The amplitude A is not determined by the linear analysis.

9 Critical load and number of half-waves

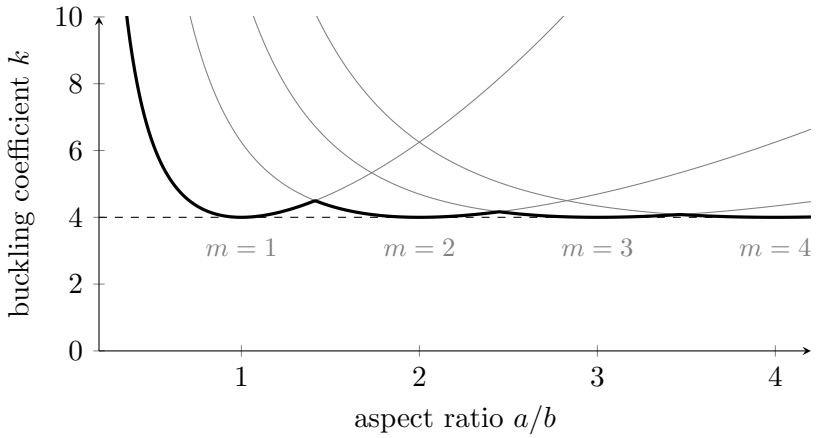
The stress (24) increases with n, so the lowest value has one half-wave across the width, $n = 1$:

$$\sigma = k\,\frac{\pi^2 D}{b^2 t}, \qquad k = \left(\frac{mb}{a} + \frac{a}{mb}\right)^2. \tag{25}$$

The plate buckles at the integer m that gives the smallest k. Each value of m gives one curve of k against a/b, with a minimum $k = 4$ at $a/b = m$. Two neighbouring curves cross where $k_m = k_{m+1}$, which gives

$$\frac{a}{b} = \sqrt{m(m+1)}. \tag{26}$$

The number of half-waves therefore changes from 1 to 2 at $a/b = \sqrt{2} \approx 1.41$, from 2 to 3 at $\sqrt{6} \approx 2.45$, and so on. For a long plate, k stays close to 4 and each half-wave is close to square ($a/m \approx b$).



Critical stress and critical load. With $D = Et^3/[12(1-\nu^2)]$,

$$\sigma_{cr} = k\,\frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2, \qquad P_{cr} = \sigma_{cr}\, b\, t. \tag{27}$$

The plate stays flat for $P < P_{cr}$ and can take the buckled shape for $P \geq P_{cr}$.

10 Example

An aluminium plate with $E = 70$ GPa, $\nu = 0.33$, $a = 600$ mm, $b = 200$ mm and $t = 2$ mm:

- $D = \dfrac{70 \times 10^9 \times (0.002)^3}{12\,(1 - 0.33^2)} = 52.4$ N m.
- $a/b = 3$, so $m = 3$ and $k = (1+1)^2 = 4$.
- $\sigma_{cr} = \dfrac{4\,\pi^2 \times 52.4}{(0.2)^2 \times 0.002} = 25.8$ MPa.
- $P_{cr} = 25.8 \times 10^6 \times 0.2 \times 0.002 = 10.3$ kN.

These are the default values of the plate buckling applet.

11 Remarks

1. **Comparison with a column.** If the two unloaded edges were free instead of simply supported, the plate would bend into one half-wave along a with no curvature across the width. It would then behave approximately as a wide column, with $\sigma_{cr} \approx \pi^2 D/(a^2 t)$, which is the Euler result with EI replaced by D per unit width. Supporting the unloaded edges forces the half-waves to be about as long as b, which raises the critical stress.

2. **Other edge conditions.** For long plates, the coefficient k depends on the supports of the unloaded edges, for example:

Unloaded edges	k (long plate)
both simply supported	4.00
both clamped	6.97
one simply supported, one free	0.425

3. **After buckling.** The linear analysis gives the critical load but not the amplitude. A real plate continues to carry increasing load after buckling, because its deflection stretches the mid-surface; this is described by the nonlinear equations (18) and (19). The column, by contrast, carries almost no additional load after buckling, and the axially compressed cylinder carries less.