# The Shear Beam

Companion to the interactive simulator at https://aimlab.uk/sims/shear-beam.html

## Setup and assumptions

A cantilever of length $L$ and rectangular cross-section $w \times h$ carries a transverse tip load $F$; the material has shear modulus $G$. The idealisation is opposite to Euler–Bernoulli: *no bending stiffness*, deformation by shear alone — cross-sections translate laterally by $u(x)$ without rotating.

## Kinematics

With sections sliding but not rotating, the change of angle between the beam axis and the cross-section is carried entirely by shear: $\gamma = \mathrm{d}u/\mathrm{d}x$, taken as uniform over the section in this idealisation.

## Constitutive law and the correction factor

The shear force is the resultant of the shear stress over the section. In reality $\tau$ is not uniform — for a rectangle it is parabolic,

$$\tau(y) \;=\; \frac{3V}{2A}\left(1-\left(\frac{2y}{h}\right)^2\right), \qquad \tau_{\max} \;=\; \frac{3V}{2A} \quad \text{at mid-depth,} \tag{1}$$

vanishing on the free surfaces. Matching the shear strain energy of this distribution to that of the uniform model gives the correction factor $k = 5/6$ for a rectangle. The section law is then

$$V \;=\; kGA\,\gamma \;=\; kGA\,u', \qquad A = wh. \tag{2}$$

$kGA$ is the *shear rigidity* — it plays the role that $EI$ plays in bending.

## Governing equation

Transverse equilibrium of a slice gives $\mathrm{d}V/\mathrm{d}x = -q = 0$ between loads, hence a second-order equation in $G$ (compare the fourth-order $EI\,u'''' = 0$ of the bending beam):

$$kGA\,\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \;=\; 0, \qquad u(0) = 0, \qquad kGA\,u'(L) = F. \tag{3}$$

## Solution

$$\boxed{u(x) \;=\; \frac{Fx}{kGA}} \qquad \Rightarrow \qquad \delta \;=\; \frac{FL}{kGA}, \qquad \gamma \;=\; \frac{F}{kGA} \;\text{(constant).} \tag{4}$$

The deflected shape is a straight line with a kink at the wall.

## When does shear matter?

For the same beam, the Euler–Bernoulli tip deflection is $FL^3/3EI$. The ratio of the two is

$$\frac{\delta_{\text{shear}}}{\delta_{\text{bend}}} \;=\; \frac{3EI}{kGAL^2} \;=\; \frac{E}{4kG}\left(\frac{h}{L}\right)^2, \tag{5}$$

which scales with $(h/L)^2$: negligible for slender beams, significant for stubby ones. Physical beams do both, which is captured by the Timoshenko beam formulation.

## Worked example

$G = 1$ MPa, $L = 500$ mm, $w = 25$ mm, $h = 50$ mm, $F = 60$ N:

$$A = 1250\ \text{mm}^2, \qquad kGA = \tfrac{5}{6} \times 10^6 \times 1.25 \times 10^{-3} = 1042\ \text{N}, \qquad \delta = \frac{60 \times 0.5}{1042} = 28.8\ \text{mm}, \tag{6}$$

with $\gamma = 5.8\%$ and $\tau_{\max} = 3F/2A = 72$ kPa. Set the simulator sliders to these values and compare.