SOLID MECHANICS

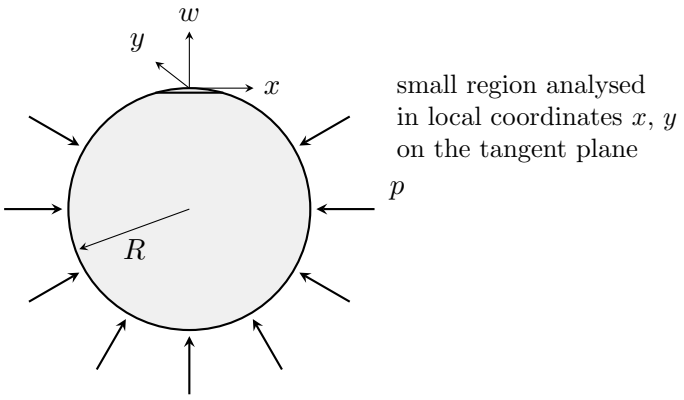
Sphere buckling: derivation

Buckling of a complete thin spherical shell under uniform external pressure

1 The problem

A complete spherical shell of radius R (measured to the mid-surface) and wall thickness t carries a uniform external pressure p. Before buckling, the shell stays spherical and every direction in the wall carries the same compressive force per unit length,

$$N = -\frac{pR}{2}, \qquad \sigma = \frac{pR}{2t} \qquad \text{(compression positive)}. \tag{1}$$



Near any point of the sphere, x and y are Cartesian coordinates on the tangent plane, z is measured through the wall from the mid-surface, and $w(x, y)$ is the radial displacement, positive outward. The external pressure acts inward.

Assumptions.

- The wall is thin (t much smaller than R), and normals to the mid-surface remain straight and normal to it (Kirchhoff–Love kinematics).
- The material is linear elastic and isotropic, with Young's modulus E and Poisson's ratio ν.
- The buckle is small compared with the radius, so a region of the sphere can be treated as a shallow shell. The curvature $1/R$ of the sphere then enters only through the terms w/R in the strains and N/R in the equilibrium equation.
- Rotations are moderate: only the slopes $\partial w/\partial x$ and $\partial w/\partial y$ enter the nonlinear terms.
- The change in the pressure load caused by the buckling deformation is neglected.

The derivation has three parts, as for the column, the plate and the cylinder: kinematics, compatibility and equilibrium. The material law then combines them.

2 Kinematics

Membrane strains. An outward displacement w moves the wall to a sphere of radius $R + w$, which lengthens every line element by the fraction w/R, in both directions:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}, \quad (2)$$

where u and v are the in-plane displacements of the mid-surface.

Changes of curvature, as for a flat plate (shallow-shell approximation):

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \qquad \kappa_{xy} = -\frac{\partial^2 w}{\partial x\,\partial y}. \quad (3)$$

For a flat plate, the terms w/R are absent. For a cylinder, w/R appears only in the hoop strain ε_y, because the wall is curved in one direction only.

3 Compatibility

The three membrane strains in (2) come from only two displacements, u and v. Combining their second derivatives so that u and v cancel leaves

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\,\nabla^2 w, \quad (4)$$

with $\nabla^2 w = \partial^2 w/\partial x^2 + \partial^2 w/\partial y^2$. The last term comes from the two terms w/R, and it is linear in w: any buckle that bends the wall also stretches it.

For a flat plate, the term with $1/R$ is absent. For a cylinder, it is $(1/R)\,\partial^2 w/\partial x^2$.

4 Equilibrium

In-plane force balance.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0. \quad (5)$$

Force balance normal to the wall (outward positive). Moment balance gives the transverse shear forces, as for the plate. The in-plane forces acting on the deflected wall, and on the curvature $1/R$ of the sphere in both directions, add normal components:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x\,\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\,\frac{\partial^2 w}{\partial x\,\partial y} + N_y\,\frac{\partial^2 w}{\partial y^2} - \frac{N_x + N_y}{R} - p = 0. \quad (6)$$

Check with the state before buckling. With w uniform, the moments and the slope terms vanish, and (6) gives $-2N/R - p = 0$, so $N = -pR/2$, as stated in Section 1.

For a flat plate, the terms with $1/R$ are absent. For a cylinder, only $-N_y/R$ appears.

5 Material law and governing equations

For a thin wall in plane stress, the membrane forces and moments are

$$N_x = \frac{Et}{1-\nu^2}\,(\varepsilon_x + \nu\varepsilon_y), \quad N_y = \frac{Et}{1-\nu^2}\,(\varepsilon_y + \nu\varepsilon_x), \quad N_{xy} = \frac{Et}{2(1+\nu)}\,\gamma_{xy}, \tag{7}$$

$$M_x = D\,(\kappa_x + \nu\kappa_y), \quad M_y = D\,(\kappa_y + \nu\kappa_x), \quad M_{xy} = D\,(1-\nu)\,\kappa_{xy}, \tag{8}$$

with $D = Et^3/[12(1-\nu^2)]$. The moment terms in (6) become $-D\nabla^4 w$.

Stress function. The in-plane equations (5) are satisfied exactly by $N_x = \partial^2 F/\partial y^2$, $N_y = \partial^2 F/\partial x^2$ and $N_{xy} = -\partial^2 F/\partial x\,\partial y$, and the left-hand side of (4) then equals $\nabla^4 F/(Et)$.

Governing equations (shallow spherical shell).

$$D\,\nabla^4 w = \frac{\partial^2 F}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \frac{1}{R}\,\nabla^2 F - p, \tag{9}$$

$$\frac{1}{Et}\,\nabla^4 F = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \frac{1}{R}\,\nabla^2 w. \tag{10}$$

State before buckling. $F_0 = -\frac{1}{4}\,pR\,(x^2+y^2)$ gives $N_x = N_y = -pR/2$ and $N_{xy} = 0$. The membrane strain is $(1-\nu)N/(Et)$ in every direction, and with $u = v = 0$ by symmetry, the shell shrinks uniformly by

$$w_0 = -\frac{(1-\nu)\,pR^2}{2Et}. \tag{11}$$

Both (9) and (10) are satisfied by F_0 and w_0.

6 Linearisation for buckling

Write $w = w_0 + w_1$ and $F = F_0 + f$, where w_1 and f describe the buckle, and keep only the terms linear in w_1 and f:

$$D\,\nabla^4 w_1 + \frac{pR}{2}\,\nabla^2 w_1 + \frac{1}{R}\,\nabla^2 f = 0, \qquad \nabla^4 f = \frac{Et}{R}\,\nabla^2 w_1. \tag{12}$$

Unlike the plate equations, these two equations remain coupled through the terms with $1/R$.

7 Boundary conditions

A complete sphere has no edges, so no boundary conditions apply. The local analysis requires the buckle to be small compared with the radius; Section 9 shows that the buckle size is of order $\sqrt{Rt}$, which is much smaller than R when t is much smaller than R.

8 Solution

Trial solution. A wave of any direction on the tangent plane,

$$w_1 = A\cos(k_x x + k_y y), \qquad k^2 = k_x^2 + k_y^2, \tag{13}$$

gives $\nabla^2 w_1 = -k^2 w_1$ and $\nabla^4 w_1 = k^4 w_1$. The compatibility equation in (12) then gives $f = -Et\, w_1/(Rk^2)$.

Substitution into the first equation of (12):

$$\left[Dk^4 - \frac{pR}{2}\,k^2 + \frac{Et}{R^2}\right] A = 0. \tag{14}$$

Eigenvalue condition. A non-zero A requires

$$\frac{pR}{2} = Dk^2 + \frac{Et}{R^2 k^2}. \tag{15}$$

This is the same relation as for the axisymmetric buckle of a cylinder, with σt replaced by $pR/2$. The first term is the bending resistance, and the second the resistance from stretching of the curved wall. As for the cylinder, for a given pressure a real wave number k exists only when p is at least the critical value below.

9 Critical pressure and buckle size

Minimising (15) over k gives

$$k_c^4 = \frac{Et}{DR^2}, \qquad p_{cr} = \frac{4\sqrt{D\,Et}}{R^2} = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{R}\right)^2 \approx 1.21\,E\left(\frac{t}{R}\right)^2 \quad (\nu = 0.3). \tag{16}$$

The corresponding membrane stress is the same as the critical stress of the axially compressed cylinder:

$$\sigma_{cr} = \frac{p_{cr} R}{2t} = \frac{E\,t}{R\sqrt{3(1-\nu^2)}}. \tag{17}$$

The half-wavelength of the buckle is $\pi/k_c = \pi\sqrt{Rt}/[12(1-\nu^2)]^{1/4} \approx 1.73\sqrt{Rt}$ for $\nu = 0.3$. The shell stays spherical for $p < p_{cr}$ and can take a buckled shape for $p \geq p_{cr}$.

Many modes at one pressure. Only the magnitude $k = k_c$ is fixed; the direction of the wave is arbitrary, and any superposition of such waves is also a mode. On the complete sphere, the modes are spherical harmonics of degree n, for which ∇^2 has the eigenvalue $-n(n+1)/R^2$. The critical degree therefore satisfies

$$n(n+1) \approx k_c^2 R^2 = 2\sqrt{3(1-\nu^2)}\,\frac{R}{t}, \tag{18}$$

and all $2n+1$ harmonics of that degree buckle at the same pressure. The applet draws the axisymmetric one, $w_1 \propto P_n(\cos\phi)$, where P_n is the Legendre polynomial and ϕ the angle from a pole. Zoelly's analysis of the complete sphere with spherical harmonics (thesis, Zürich, 1915) gives the same critical pressure for thin shells.

10 Example

An aluminium sphere with $E = 70$ GPa, $\nu = 0.3$, $R = 500$ mm and $t = 2$ mm:

- $R/t = 250$ and $\sqrt{3(1-\nu^2)} = 1.652$.

- $p_{cr} = \dfrac{2 \times 70 \times 10^9}{1.652} \times \left(\dfrac{2}{500}\right)^2 = 1.36$ MPa.
- $\sigma_{cr} = \dfrac{1.36 \times 10^6 \times 0.5}{2 \times 0.002} = 169$ MPa, which is below the yield stress of common structural aluminium alloys such as 6061-T6.
- Half-wavelength $= 1.73\sqrt{500 \times 2}$ mm $= 54.7$ mm.
- $n(n+1) \approx 3.30 \times 250 = 826$, so $n = 28$.

These are the default values of the sphere buckling applet.

11 Remarks

1. **Imperfection sensitivity.** The pressure falls steeply after buckling, so small departures from the spherical shape lower the buckling pressure substantially. Spherical shells are among the most imperfection-sensitive structures: tests give buckling pressures well below p_{cr}, with wide scatter.
2. **Clamped shallow caps.** For a cap of finite rise clamped at its edge, the buckling pressure depends on a single geometric parameter that combines the rise of the cap and the wall thickness, and it has to be found numerically. For very shallow caps it is well below p_{cr}; for deeper caps it is of the same order as p_{cr}.
3. **Comparison with the cylinder.** The sphere and the axially compressed cylinder have the same critical membrane stress and the same buckle size. On the cylinder, the critical wave numbers lie on the Koiter circle; on the sphere, any direction of the wave is critical.
4. **Yield.** If σ_{cr} exceeds the yield stress, the shell yields before it buckles elastically, and this result does not apply.