# Torsion of a Circular Shaft

Companion to the interactive simulator at https://aimlab.uk/sims/torsion.html

## Setup and assumptions

A solid circular shaft of diameter $d$ and length $L$ is fixed at $x = 0$ and twisted by a torque $T$ at $x = L$. The material is linear elastic with shear modulus $G$; rotations are small. Let $\varphi(x)$ be the rotation of the cross-section at $x$.

## Kinematics

By circular symmetry, every cross-section rotates rigidly in its own plane — plane sections remain plane and undistorted. This is true only for circular sections; any other shape warps. Consider two neighbouring sections a distance $\mathrm{d}x$ apart, rotated relative to each other by $\mathrm{d}\varphi$. A material line of length $\mathrm{d}x$ on the cylinder of radius $r$ tilts by the angle

$$\gamma(r) \;=\; \frac{r\,\mathrm{d}\varphi}{\mathrm{d}x} \;=\; r\,\varphi', \tag{1}$$

which is the shear strain: zero on the axis, largest at the surface.

## Constitutive law and stress resultant

Hooke's law in shear gives $\tau(r) = G\gamma(r) = G\,r\,\varphi'$. The torque carried by the section is the moment of this stress distribution:

$$T(x) \;=\; \int_A \tau\,r\,\mathrm{d}A \;=\; G\varphi' \int_A r^2\,\mathrm{d}A \;=\; GJ\varphi', \qquad J \;=\; \int_0^{d/2} r^2\,(2\pi r)\,\mathrm{d}r \;=\; \frac{\pi d^4}{32}. \tag{2}$$

$J$ is the polar second moment of area — the torsional analogue of $I$.

## Governing equation

With no distributed torque along the shaft, equilibrium of a slice gives $\mathrm{d}T/\mathrm{d}x = 0$, so

$$GJ\,\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} \;=\; 0, \qquad \varphi(0) = 0, \qquad GJ\,\varphi'(L) = T. \tag{3}$$

Exactly the same second-order structure as the axial rod, with $(u, EA, F) \to (\varphi, GJ, T)$.

## Solution

$$\boxed{\varphi(x) \;=\; \frac{Tx}{GJ}} \qquad \Rightarrow \qquad \varphi_L \;=\; \frac{TL}{GJ}, \qquad \tau_{\max} \;=\; \frac{T\,(d/2)}{J} \;=\; \frac{16\,T}{\pi d^3}, \qquad \gamma_{\max} \;=\; \frac{(d/2)\,\varphi_L}{L}. \tag{4}$$

Note that the twist depends on $G$ but the stress does not: change the material in the simulator and watch $\varphi$ collapse while $\tau_{\max}$ stays put. Since $J \propto d^4$, doubling the diameter makes the shaft sixteen times stiffer in twist.

## Worked example (simulator defaults)

$G = 10$ MPa, $d = 20$ mm, $L = 300$ mm, $T = 0.5$ N m:

$$J = \frac{\pi \times 20^4}{32} = 15\,708\ \mathrm{mm}^4, \qquad \varphi_L = \frac{0.5 \times 0.3}{10^7 \times 1.571 \times 10^{-8}} = 0.955\ \mathrm{rad} = 54.7^\circ, \tag{5}$$

with $\tau_{\max} = 0.318$ MPa and $\gamma_{\max} = 3.18\%$. Set the simulator sliders to these values and compare the dial. Beyond roughly 10% surface shear strain the small-strain assumption becomes invalid.